



Lecture 13 Deutsch algorithm

Deutsch algorithm recipe

optical implementation of Deutsch algorithm

Assuming $\hat{U}_f = \hat{U}_f^\dagger$ (unitary)

$x \rightarrow \hat{U}_f \rightarrow |f(x)\rangle |x\rangle$
 $f(a) \rightarrow \hat{U}_f \rightarrow |f(a)\rangle |a\rangle$
 $f(b) \rightarrow \hat{U}_f \rightarrow |f(b)\rangle |b\rangle$

$a, b \rightarrow f_1 \rightarrow f(a) = f(b)$
 $a \neq b \rightarrow f_1 \rightarrow f(a) \neq f(b)$

$|a\rangle \xrightarrow{f_1} |a\rangle |f(a)\rangle$
 $|b\rangle \xrightarrow{f_1} |b\rangle |f(b)\rangle$
 $(f(a) = f(b))$

②

$|x\rangle \rightarrow \hat{U}_f \rightarrow |x\rangle$
 $|y\rangle \rightarrow \hat{U}_f \rightarrow |f(y)\rangle$

③ $|x\rangle |y\rangle \xrightarrow{\hat{U}_f} |x\rangle |f(y)\rangle$

If $|y\rangle = |0\rangle$
 then $|x\rangle |0\rangle \xrightarrow{\hat{U}_f} |x\rangle |f(x)\oplus 0\rangle = |x\rangle |f(x)\rangle$
 $|x\rangle |0\rangle \xrightarrow{f} |x\rangle |f(x)\oplus 0\rangle = |x\rangle |f(x)\rangle$
 $|x\rangle |f(x)\rangle \xrightarrow{U_f} |x\rangle |f(x)\oplus f(x)\rangle = |x\rangle |0\rangle \rightarrow \hat{U}_f = \hat{U}_f^\dagger$

If $|y\rangle = |1\rangle$
 then $|x\rangle |1\rangle \xrightarrow{\hat{U}_f} |x\rangle |f(x)\oplus 1\rangle$
 $|x\rangle |f(x)\rangle \xrightarrow{U_f} |x\rangle |f(x)\oplus f(x)\rangle = |x\rangle |0\rangle$

Therefore, $\hat{U}_f = \hat{U}_f^\dagger$

Note

$|a \oplus 0\rangle = |a\rangle$
 $|a \oplus 1\rangle = |\bar{a}\rangle$
 $|a \oplus a\rangle = |0\rangle$
 $|a \oplus \bar{a}\rangle = |1\rangle$

$a \oplus b = \text{mod}(a+b, 2)$
 $= R\left(\frac{a+b}{2}\right)$
 $= 1, 0$





- Deutsch algorithm recipe

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$\{1, 2, 3, 4, 5, \dots\} \xrightarrow{f} \{0, 1\}$

③ Not all gates are Hermitian (but they are all unitary)
→ quantum processor (made of many gates) will be unitary
and we require it also to be Hermitian, so that the quantum processor is reversible.

② $U_f = U_k \dots U_2 U_1 U_0 U_k$

$\langle \phi | \phi \rangle = 1 \rightarrow \langle \phi | \phi \rangle = 1$

$\langle \phi | U_0^\dagger U_1^\dagger \dots U_k^\dagger | \phi \rangle = 1$

$U_0^\dagger U_1^\dagger \dots U_k^\dagger = I$

$U_0^\dagger = U_0^{-1}$ (unitary)
 $U_1^\dagger = U_1^{-1}$ (unitary)
 $U_k^\dagger = U_k^{-1}$ (unitary)

$U = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$
 $U^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$
 $U^\dagger U = I$ (unitary)
 $U^\dagger = U^{-1}$ (Hermitian)

④

$\begin{bmatrix} |0\rangle \\ |1\rangle \end{bmatrix} \rightarrow \{0, 1\}$

$\begin{bmatrix} |0\rangle|0\rangle \\ |0\rangle|1\rangle \\ |1\rangle|0\rangle \\ |1\rangle|1\rangle \end{bmatrix} \rightarrow \{0, 1\}^2$

$\{00, 01, 10, 11\}$
 $0 \ 1 \ 2 \ 3$

$|0\rangle \downarrow 2 \rightarrow 10$
 $|1\rangle \downarrow 3 \rightarrow 11$
 $|0\rangle + |1\rangle \downarrow 5 \rightarrow 101$
 $|0\rangle - |1\rangle \downarrow 6 \rightarrow 110$
 $|0\rangle + i|1\rangle \downarrow 7 \rightarrow 10i$
 $|0\rangle - i|1\rangle \downarrow 8 \rightarrow 10(-i)$

$2+3=5$
 $2+3=8$
 $6+7=13$

$1010 \leftarrow \{0, 1\}^4$





(5) Deutsch algorithm
Deutsch-Jozsa Algorithm.

$\{0, 1\}$
 $\{0, 1\}^n$

$f(x) = c$

$f(0) = 1$
 $f(1) = 1$
 $f(2) = 1$
 $f(4) = 1$

$|0\rangle = |00\rangle \xrightarrow{f} 3$
 $|1\rangle = |01\rangle \xrightarrow{f} 2$
 $|2\rangle = |10\rangle \xrightarrow{f} 1$
 $|3\rangle = |11\rangle \xrightarrow{f} 0$

100
 $110100 \leftarrow \{1, 0\}^6$
 $2^6 = 64$

Deutsch Construction.

unchanged ($|0\rangle$) means constant
changed ($|1\rangle$) means balanced

$|0\rangle = |0\rangle$
 $|1\rangle = |1\rangle$

A $f(0) = f(1) = 0$
 $|1_0\rangle = \frac{1}{\sqrt{2}} [|0\rangle_1 |0\rangle_2 - |1\rangle_1 |1\rangle_2]$
 $+ |1\rangle_1 (|0\rangle_2 - |1\rangle_2)$
 $= \frac{1}{2} [(|0\rangle_1 + |1\rangle_1) (|0\rangle_2 - |1\rangle_2)]$
 $|1_1\rangle = \frac{1}{2} [|0\rangle_1 (|0\rangle_2 - |1\rangle_2)]$
detect: $|0\rangle_1$

B $f(0) = f(1) = 1$
 $\frac{1}{\sqrt{2}} [|0\rangle_1 (|1\rangle_2 - |0\rangle_2)]$
 $+ |1\rangle_1 (|1\rangle_2 - |0\rangle_2)$
 $= \frac{1}{2} [(|0\rangle_1 + |1\rangle_1) (|1\rangle_2 - |0\rangle_2)]$
 $\times (|1\rangle_2 - |0\rangle_2)$

C $f(0) = 0, f(1) = 1$
 $\frac{1}{\sqrt{2}} [|0\rangle_1 (|0\rangle_2 - |1\rangle_2)]$
 $+ |1\rangle_1 (|1\rangle_2 - |0\rangle_2)$
 $= \frac{1}{2} [(|0\rangle_1 + |1\rangle_1) (|0\rangle_2 - |1\rangle_2)]$
 $+ \frac{1}{2} [(|0\rangle_1 - |1\rangle_1) (|1\rangle_2 - |0\rangle_2)]$
 $= \frac{1}{2} [|0\rangle_1 (|0\rangle_2 - |1\rangle_2) - |1\rangle_1 (|1\rangle_2 - |0\rangle_2)]$
 $= \frac{1}{2} [|0\rangle_1 (|0\rangle_2 - |1\rangle_2) - |1\rangle_1 (|1\rangle_2 - |0\rangle_2)]$

D $f(0) = 1, f(1) = 0$
 $\frac{1}{\sqrt{2}} [|1\rangle_1 (|0\rangle_2 - |1\rangle_2)]$
 $+ |0\rangle_1 (|1\rangle_2 - |0\rangle_2)$
 $= \frac{1}{2} [|1\rangle_1 (|0\rangle_2 - |1\rangle_2) + |0\rangle_1 (|1\rangle_2 - |0\rangle_2)]$
 $= \frac{1}{2} [|1\rangle_1 (|0\rangle_2 - |1\rangle_2) + |0\rangle_1 (|1\rangle_2 - |0\rangle_2)]$
 $= \frac{1}{2} [|1\rangle_1 (|0\rangle_2 - |1\rangle_2) + |0\rangle_1 (|1\rangle_2 - |0\rangle_2)]$

$A = A^\dagger$





①

$$\begin{aligned}
 & |0\rangle|1\rangle \xrightarrow{H_1, H_2} \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \\
 &= \frac{1}{2} \left[|0\rangle(|0\rangle - |1\rangle) + |1\rangle(|0\rangle - |1\rangle) \right] \\
 &\xrightarrow{U_f} \frac{1}{2} \left[|0\rangle(|f(0)\oplus 0\rangle - |f(0)\oplus 1\rangle) \right. \\
 &\quad \left. + |1\rangle(|f(1)\oplus 0\rangle - |f(1)\oplus 1\rangle) \right] \\
 &= \frac{1}{2} \left[|0\rangle(|f(0)\rangle - |f(0)\rangle) \right. \\
 &\quad \left. + |1\rangle(|f(1)\rangle - |f(1)\rangle) \right] \quad \text{--- ①} \\
 &\text{① } f(0)=f(1) \text{ (constant)} \quad \text{② } f(0)+f(1) \rightarrow f(1)=f(0) \\
 &\xrightarrow{H_2} \frac{1}{2} \left[|0\rangle(|f(0)\rangle - |f(0)\rangle) \right. \\
 &\quad \left. + |1\rangle(|f(0)\rangle - |f(0)\rangle) \right] \quad \text{①} \rightarrow \frac{1}{2} \left[|0\rangle(|f(0)\rangle - |f(0)\rangle) \right. \\
 &\quad \left. + |1\rangle(|f(0)\rangle - |f(0)\rangle) \right] \\
 &= \frac{1}{2} \left[|0\rangle + |1\rangle \right] \frac{1}{\sqrt{2}} (|f(0)\rangle - |f(0)\rangle) \quad \text{②} \rightarrow \frac{1}{2} \left[|0\rangle + |1\rangle \right] \frac{1}{\sqrt{2}} (|f(0)\rangle - |f(0)\rangle) \\
 &\xrightarrow{H_1} \frac{1}{2} \left[|0\rangle + |1\rangle \right] \frac{1}{\sqrt{2}} (|f(0)\rangle - |f(0)\rangle) \quad \text{--- ②}
 \end{aligned}$$

Deutsch-Jozsa algorithm
(construction)

③

$\{0, 1\}^n \rightarrow \{0, 1\}$
 $\{00, 01, 10, 11\}$

problem

① $f(x) = \text{constant}$
② $f(00) = f(10) = 0$
e.g. $f(11) = f(01) = 1$

Quantum circuit diagram showing three input qubits initialized to $|0\rangle, |0\rangle, |1\rangle$. The first two qubits pass through Hadamard gates. The third qubit passes through a Hadamard gate and then a controlled operation U_f that depends on the first two qubits. The circuit ends with Hadamard gates on the first two qubits and a measurement on the third qubit.

$$\begin{aligned}
 & |0\rangle|0\rangle|1\rangle \xrightarrow{H_1, H_2, H_3} \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \\
 &= \frac{1}{\sqrt{2}} (|00\rangle + |01\rangle + |10\rangle + |11\rangle) \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)
 \end{aligned}$$





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